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Convergent Yang-Mills matrix theories

Peter Austing

Department of Physics, University of Oxford Theoretical Physics
1 Keble Road, Oxford OX1 3NP, UK
E-mail: p.austing@physics.ox.ac.uk

John F. Wheater

Department of Physics, University of Oxford Theoretical Physics
1 Keble Road, Oxford OX1 3NP, UK
E-mail: j.wheater@physics.ox.ac.uk

Abstract: We consider the partition function and correlation functions in the bosonic and supersymmetric Yang-Mills matrix models with compact semi-simple gauge group. In the supersymmetric case, we show that the partition function converges when $D = 4, 6$ and 10, and that correlation functions of degree $k < k_c = 2(D-3)$ are convergent independently of the group. In the bosonic case we show that the partition function is convergent when $D \geq D_c$, and that correlation functions of degree $k < k_c$ are convergent, and calculate D_c and k_c for each group, thus extending our previous results for $\mathrm{SU}(N)$. As a special case these results establish that the partition function and a set of correlation functions in the IKKT IIB string matrix model are convergent.

Keywords: Nonperturbative Effects, M(atrix) Theories, Matrix Models.

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Contents

1. Introduction

The quantum mechanics obtained by dimensional reduction of Yang-Mills field theories to one dimension was first studied in the 1980s [1] and describes the sector of the original field theory in which the fields are independent of the spatial coordinates. A little later it was discovered that the supersymmetric theory (SSYM) in ten dimensions with gauge group $\mathrm{SU}(N)$ could be regarded in the large N limit as a light-cone regularization of the super-membrane [2] but it was soon realised that this theory has a continuous spectrum starting from zero energy [3]. This is in contrast to the case of strings where the spectrum is discrete, leading to a tower of states some of which are massless and therefore candidates for massless and low mass particles in the real world and the rest of which have masses proportional to the string scale which is somewhere up in the region of the Planck mass. Thus it appeared that the absence of a gap rendered the supermembrane theory useless as a phenomenological description of nature and the subject went quiet for many years.

In the ensuing period many remarkable discoveries were made in string theory. All the known super-string theories have been found to be related to each other by various duality transformations and to 11-dimensional supergravity by compactification (see [4, 5, 6] for recent reviews and further references), and the role of D-branes as solitons in string theories opened another window on non-perturbative string physics [7]. The existence of such relationships between these various string theories clearly implies that they are all different perturbative limits of an overarching theory, christened M-theory. We know M-theory must be there but we do not know how to write it down. In particular the 11-dimensional supergravity is a classical field theory for which a consistent quantization is not known; however

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it is contained in the supermembrane theory and this leads to the BFSS conjecture [8] that the regularized supermembrane theory provided by the large N limit of SSYM quantum mechanics, now christened M(atrix) Theory, does indeed represent M-theory in the light-cone gauge. Simultaneously IKKT proposed that the dimensional reduction of $D = 10$ SU(N) SSYM to zero dimensions (which is of course the reduction of the quantum mechanics by its remaining dimension) described in the large N limit the type-IIB superstring [9]. For reviews of these subjects see for example [10, 11]. These developments have generated renewed interest in Yang-Mills quantum mechanics [12]–[17]. The continuous spectrum is no longer a problem but a virtue because this is a theory of everything and must describe multi-particle states. However the number of zero energy bosonic states in M(atrix) Theory must be one because there is only supposed to be one graviton super-multiplet in the theory. It is conjectured that this is indeed the case for SU(N) [18] and calculations in the special case of SU(2) [12, 13] agree but, although there are several partial results [19]–[22], it remains otherwise unproven. The calculation of the Witten index requires the evaluation of, among other things, quantities that are partition functions in the IKKT model; these are often called Yang-Mills matrix integrals. Even here there are many potential difficulties, the most basic of which is whether the partition function and correlation functions exist and this is the main subject of this paper.

The Yang-Mills matrix integral partition function, which is obtained by dimensionally reducing the euclidean SSYM action from D down to zero dimensions, is given by

$$\mathcal{Z}_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \prod_{\alpha=1}^{\mathcal{N}} d\psi_\alpha \exp\left(\sum_{\mu,\nu} \mathrm{Tr}\,[X_\mu, X_\nu]^2 + \mathrm{Tr}\,\psi_\alpha [\Gamma^\mu_{\alpha\beta} X_\mu, \psi_\beta] \right), \tag{1.1}$$

where we adopt the summation convention for repeated indices. The traceless hermitian matrix fields X_μ and ψ_α (respectively bosonic and fermionic) are in the Lie algebra $\mathcal{G}$ of the (compact semi-simple) gauge group G and can be written

$$X_\mu = \sum_{a=1}^{g} X^a_\mu t^a , \qquad \psi_\alpha = \sum_{a=1}^{g} \psi^a_\alpha t^a , \tag{1.2}$$

where $\{t^a, a = 1, \ldots, g\}$ are the generators in the fundamental representation. The $\Gamma^\mu_{\alpha\beta}$ are ordinary gamma matrices for D euclidean dimensions. The model possesses a gauge symmetry

$$X_\mu \to U^\dagger X_\mu U , \qquad \psi_\alpha \to U^\dagger \psi_\alpha U , \qquad U \in G , \tag{1.3}$$

and an SO(D) symmetry inherited from the original D-dimensional euclidean symmetry of the SSYM. Although the motivation discussed above leads to a study of the $D = 10$, SU(N) supersymmetric integral, it is useful and illuminating to study several different versions of the model. Firstly by suppressing the fermions we get the bosonic integrals which we will denote by $\mathcal{N} = 0$ (ie there are no super-charges) [23]. Secondly the supersymmetric integrals can be written for $D = 3$, 4, 6, and 10, having $\mathcal{N} = 2(D-2)$ super-charges. In principle one can integrate out the fermions to obtain

$$\mathcal{Z}_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \, \mathcal{P}_{D,G}(X_\mu) \exp\left(\sum_{\mu,\nu} \mathrm{Tr}\,[X_\mu, X_\nu]^2 \right), \tag{1.4}$$

where the pfaffian $\mathcal{P}_{D,G}$ is a homogeneous polynomial of degree $\frac{1}{2}\mathcal{N}g$. We will also consider simple correlation functions of the form

$$\langle \mathcal{C}_k(X_\sigma) \rangle = \int \prod_{\mu=1}^{D} dX_\mu \, \mathcal{C}_k(X_\sigma) \, \mathcal{P}_{D,G}(X_\mu) \exp\left(\sum_{\mu,\nu} \mathrm{Tr}\,[X_\mu, X_\nu]^2 \right), \tag{1.5}$$

where $\mathcal{C}_k$ is a polynomial of degree k.

Only when the gauge group is SU(2) is it known how to evaluate all these integrals in closed analytic form [1, 12, 13], [23]–[26]. We recently established analytically the convergence criteria for the bosonic integrals in the case of SU(N) [27] and in this paper we will extend these results to all other compact semi-simple gauge groups and to the supersymmetric integrals. However much has already been learned about the properties of these integrals by a variety of techniques. The authors of [20] used the supersymmetry to deform the Yang-Mills partition function into a cohomological theory in which the integrals can be done. From the point of view of the defining formula (1.4) this involves among other things a change of variables and an analytic continuation that implicitly depends upon the original integral being convergent. The method appears to work for the partition function when $D > 3$ and gives results in agreement with numerical calculations at small N [28, 23, 17]. Unfortunately it seems that this method cannot be used to calculate the many correlation functions that are of interest in the original YM model. There is no small parameter expansion in these models (the original gauge coupling g can be scaled out) but the one-loop effective action can be calculated as can the $1/D$ expansion [29]–[31]; care must be exercised in the gaussian approximation because it requires a cut-off to be introduced even when the full theory is actually convergent.

It was realised in [28] that these integrals are amenable to numerical calculation for small gauge groups where the pfaffian can be handled more or less by brute force; very careful and accurate determination of the integrals for SU(N) with $N = 3$, 4 and 5 [23] confirmed the results obtained by deformation (there are a number of subtleties involved in this comparison arising from the normalization of the measures, in addition to the problems of actually doing the integrals). These authors also examined the bosonic integrals, which were commonly believed to be divergent because of the flat directions in the action, and found that in fact provided D is big enough they converge too. In later papers these results were extended to other gauge

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groups [32, 17]; as we shall see the conclusions about convergence contained therein are entirely in agreement with the analytic results that we explain in this paper.

Another use of numerical simulations is to study correlation functions at much larger N by Monte Carlo and to try to establish the large N behaviour of the theory. Simulations for the SU(N) bosonic model up to $N = 768$ have now been reported [30, 33, 34]; as can be seen from the analytic bounds [27] the convergence properties at large N are very good and a great deal of information can be obtained. Intriguingly it even appears that the Wilson loop shows an area law in a regime which remains finite as N increases. It is not quite so easy to study large N for the supersymmetric theories because of the pfaffian; brute force evaluation is out of the question. When $D = 4$ the pfaffian is positive semi-definite and can be expressed as a determinant; this means that it is possible to deal with the fermions by Monte Carlo using the standard methods of lattice gauge theory and values of N up to 48 have been studied [35, 36, 33]. For $D = 6, 10$ the pfaffian causes real trouble because it is complex and standard methods do not work. Two ways of coping with this have been tried; in [37] the one loop effective action with an ultraviolet cut-off was simulated using the absolute value of the pfaffian while [38, 39] studied configurations which are saddle-points of the phase of the pfaffian. Although both of these calculations violate supersymmetry it is interesting that the latter leads to lower dimensional sub-manifolds dominating the integral whereas the former does not. The Yang-Mills quantum mechanics has also been studied in the quenched approximation [40] and a supersymmetric random surface model has been simulated directly [41] and compared to the IKKT model.

This paper has two purposes. The first is to show how to extend our convergence proofs for bosonic partition functions and correlators from SU(N) to all other compact gauge groups; in section 2 we show which integrals converge and in section 3 we conversely show which ones diverge. The second purpose is to repeat the exercise for the supersymmetric models which is done in section 4. Section 5 is a discussion of our results.

2. Convergent bosonic integrals

We consider first the integral (1.1) without fermions so that $\mathcal{N} = 0$ and there is no pfaffian. The dangerous regions which might cause (1.1) to diverge are where all the commutators almost vanish but the magnitude of X_μ goes to infinity. Hence we let

$$X_\mu = R x_\mu \,, \qquad \mathrm{Tr}\, x_\mu x_\mu = 1 \,. \tag{2.1}$$

Then

$$\mathcal{Z}_{D,G} = \int_0^\infty dR R^{Dg-1} \mathcal{X}_{D,G}(R) \,, \tag{2.2}$$

where

$$\mathcal{X}_{D,G}(R) = \int \prod_{\nu=1}^{D} dx_\nu \; \delta\left(1 - \operatorname{Tr} x_\mu x_\mu\right) \exp\left(-R^4 S_{\mathcal{G}}\right) \tag{2.3}$$

and

$$\begin{aligned} S_{\mathcal{G}} &= -\operatorname{Tr}\left[x_\mu, x_\nu\right]\left[x_\mu, x_\nu\right] \\ &= \sum_{i,j,\mu,\nu} \left| \left[x_\mu, x_\nu\right]_{i,j} \right|^2 . \end{aligned} \tag{2.4}$$

We note that for any R the integral $\mathcal{X}_{D,G}(R)$ is bounded by a constant and, if for large R

$$|\mathcal{X}_{D,G}(R)| < \frac{\text{const}}{R^\alpha}, \qquad \text{with } \alpha > Dg\,, \tag{2.5}$$

then the partition function $\mathcal{Z}_{D,G}$ is finite. Our tactic for proving convergence of $\mathcal{Z}_{D,G}$ is to find a bound of the form (2.5) on $\mathcal{X}_{D,G}(R)$. A sufficient condition for the correlation function (1.5) to converge is obtained by modifying (2.5) to require $\alpha > Dg + k$.

From now on, we are only interested in large R, so we shall always assume $R > 1$. Let us split the integration region in (2.3) into two

$$\begin{aligned} &\mathcal{R}_1(\mathcal{G}): \;\; S_{\mathcal{G}} < (R^{-(2-\eta)})^2 \\ &\mathcal{R}_2(\mathcal{G}): \;\; S_{\mathcal{G}} \geq (R^{-(2-\eta)})^2 \,, \end{aligned} \tag{2.6}$$

where η is small but positive. We see immediately that the contribution to $\mathcal{X}_{D,G}(R)$ from $\mathcal{R}_2(\mathcal{G})$ is bounded by $A_1 \exp(-R^{2\eta})$ (we will use the capital letters A, B and C to denote constants throughout this paper) and thus automatically satisfies (2.5). Thus we can confine our efforts to the contribution from $\mathcal{R}_1(\mathcal{G})$ in which we replace the exponential function by unity to get the bound

$$|\mathcal{X}_{D,G}(R)| < A_1 \exp(-R^{2\eta}) + \mathcal{I}_{D,G}(R)\,, \tag{2.7}$$

where

$$\mathcal{I}_{D,G}(R) = \int_{\mathcal{R}_1(\mathcal{G})} \prod_{\nu=1}^{D} dx_\nu \; \delta\left(1 - \operatorname{Tr} x_\mu x_\mu\right). \tag{2.8}$$

The condition in (2.1) means that at least one of the matrices x_μ (say x_1) must satisfy

$$\operatorname{Tr} x_1 x_1 \geq D^{-1}\,. \tag{2.9}$$

It is convenient to express the Lie algebra $\mathcal{G}$ using the Cartan-Weyl basis

$$\{H^i, E^\alpha\}\,, \tag{2.10}$$

where i runs from 1 to the rank l and α denotes a root. In this basis

$$\left[H^i, H^j\right] = 0\,, \qquad \left[H^i, E^\alpha\right] = \alpha^i E^\alpha \tag{2.11}$$

and

$$\begin{aligned} \left[E^{\alpha}, E^{\beta}\right] &= N_{\alpha\beta} E^{\alpha+\beta} && \text{if } \alpha+\beta \text{ is a root} \\ &= 2|\alpha|^{-2}\, \alpha \cdot H && \text{if } \alpha = -\beta \\ &= 0 && \text{otherwise.} \end{aligned} \tag{2.12}$$

Here $E^{-\alpha} = (E^{\alpha})^{\dagger}$, and the normalisation is chosen such that

$$\operatorname{Tr} H^i H^j = \delta^{ij}\,, \qquad \operatorname{Tr} E^{\alpha} E^{\beta} = 2|\alpha|^{-2}\, \delta^{\alpha+\beta}\,, \qquad \operatorname{Tr} H^i E^{\alpha} = 0\,. \tag{2.13}$$

Since the integrand and measure are gauge invariant, we can always make a gauge transformation 1.3 to move x_1 into the Cartan subalgebra

$$x_1 = x^i H^i \tag{2.14}$$

and reduce the integral over x_1 to an integral over its Cartan modes [32]

$$\prod_{a=1}^{g} dx_1^a \to \text{const} \left(\prod_{i=1}^{l} dx^i \right) \Delta_G^2(x)\,, \tag{2.15}$$

where the Weyl measure

$$\Delta_G^2(x) = \prod_{\alpha>0} (x \cdot \alpha)^2 \tag{2.16}$$

is the generalisation from $\mathrm{SU}(N)$ of the Vandermonde determinant factors. We expand the remaining x_ν

$$x_\nu = x_\nu^i H^i + x_\nu^{\alpha} E^{\alpha}\,, \qquad \nu = 2, \ldots, D \tag{2.17}$$

with $x_\nu^{-\alpha} = (x_\nu^{\alpha})^*$.

In the region $\mathcal{R}_1(\mathcal{G})$, we certainly have $-\operatorname{Tr}\,[x_1, x_\nu]^2 < R^{-2(2-\eta)}$ for $\nu = 2, \cdots, D$, and writing this in terms of the basis (2.10) gives

$$4 \sum_{\alpha>0} \frac{(x \cdot \alpha)^2}{|\alpha|^2} |x_\nu^{\alpha}|^2 < R^{-2(2-\eta)}\,, \tag{2.18}$$

where the sum is over positive roots. This is the key result because, whenever $(x \cdot \alpha)^2$ is bigger than a constant, it gives us a bound of order $R^{-(2-\eta)}$ on x_ν^{α} and so allows us to bound the integral (2.8).

There is only a finite number of ways of choosing the positive roots and it is convenient to define them in the following manner. We partition all the roots according to whether $x \cdot \alpha$ is i) positive, in which case we call α a positive root, ii) negative, in which case we call α a negative root, iii) zero, which we call the orthogonal subspace. Finally we partition the roots in the orthogonal subspace into positive or negative by the standard Cartan construction. Thus we see that if α is a positive root we have by construction $x \cdot \alpha \geq 0$. As usual there is a set of l simple positive roots $\{s_i\}$ which span the l-dimensional root space and any positive root can be be written

$$\alpha = \sum_{i=1}^{l} n_i^{\alpha} s_i \,, \tag{2.19}$$

with the $\{n_i^\alpha\}$ non-negative integers.

The integration region of x is split into a finite number of sub-regions; one for each choice of the positive roots. In each sub-region, the properties discussed in the previous paragraph hold. However, since all possible sets of simple positive roots are related by Weyl reflections, each of these sub-regions is equivalent as far as our integrals are concerned. Now define a number c

$$c = \min_{\{a^2=1\}} \max_i |a \cdot s_i| \tag{2.20}$$

which must be positive (c can be related to the quadratic form matrix but we do not need an explicit expression). Then the condition (2.9) tells us that at least one of the simple roots, s_1 say, satisfies $x \cdot s_1 \geq cD^{-1/2}$. In addition, any positive root α which contains the simple root s_1 satisfies $x \cdot \alpha \geq cD^{-1/2}$ on account of (2.19).

We now split up the Lie algebra $\mathcal{G}$ as follows. Define $\mathcal{G}' = Lie(G')$ to be the regularly embedded subalgebra of $\mathcal{G}$ obtained by omitting the simple root s_1. Then $\mathrm{rank}(\mathcal{G}') = \mathrm{rank}(\mathcal{G}) - 1$ and we write J for the combination of the generators H^i which commutes with $\mathcal{G}'$. The remaining generators are $\{F^\beta\}$ where β is any root which contains s_1. As a simple consequence of the root structure and construction of $\mathcal{G}'$, we note

$$\begin{aligned} [J, \mathcal{G}'] &= 0 \\ \left[F^\beta, \mathcal{G}'\right] &\subset \{F^\gamma\} \\ \left[J, F^\beta\right] &\subset \{F^\gamma\} \\ [\mathcal{G}', \mathcal{G}'] &\subset \mathcal{G}' \,, \end{aligned} \tag{2.21}$$

and then decompose x_μ into

$$x_\mu = y_\mu + \rho_\mu J + \omega_\mu^\beta F^\beta \,, \tag{2.22}$$

with $y_\mu \in \mathcal{G}'$; the condition (2.18) then gives us a bound on the ω_μ,

$$\left|\omega_\nu^\beta\right| < c^{-1} D^{1/2} R^{-(2-\eta)} \,, \qquad \nu = 2, \ldots, D \,. \tag{2.23}$$

We must further split up the integration region according to the relevant choice of G', and then we can use (2.23) to bound the integral (2.8) in each of these regions. The region giving the least inverse power of R will then give a bound on $\mathcal{I}_{D,G}$. Using the decomposition (2.22), the commutation rules 2.21, and the inner products (2.13), we find that the action takes the form

$$\begin{aligned} S_{\mathcal{G}}(x_\mu) &= S_{\mathcal{G}'}(y_\mu) + 2\,\mathrm{Tr}\,[y_\mu, y_\nu]\left[F^\beta, F^\gamma\right]\omega^\beta_\mu \omega^\gamma_\nu + \\ &\quad + \mathrm{Tr}\Big(\omega^\beta_\nu\left[y_\mu, F^\beta\right] - \omega^\beta_\mu\left[y_\nu, F^\beta\right] + (\rho_\mu\omega^\beta_\nu - \rho_\nu\omega^\beta_\mu)\left[J, F^\beta\right] + \omega^\beta_\mu\omega^\gamma_\nu\left[F^\beta, F^\gamma\right]\Big)^2 \\ &= S_{\mathcal{G}'}(y_\mu) + \mathcal{O}\Big(R^{-2(2-\eta)}\Big)\,, \end{aligned} \tag{2.24}$$

where we have used 2.23 and the fact that the elements of y_μ and ρ_μ are bounded by a constant. Thus we find that (up to a trivial scaling constant)

$$x_\mu \in \mathcal{R}_1(\mathcal{G}) \Rightarrow y_\mu \in \mathcal{R}_1(\mathcal{G}')\,. \tag{2.25}$$

The final ingredient is to note that the Weyl measure (2.16) for G can be bounded by that for G':

$$\Delta^2_G(x) < \mathrm{const}\,\Delta^2_{G'}(y)\,. \tag{2.26}$$

Then, integrating out the ω and ρ degrees of freedom, and using the bounds (2.23), (2.25) and (2.26) gives (more details of these manipulations are given in our previous paper [27])

$$\mathcal{I}_{D,G}(R) < B_1 R^{-(2-\eta)(D-1)(g-g'-1)}\mathcal{F}_{D,G'}(R)\,, \tag{2.27}$$

where

$$\mathcal{F}_{D,G'}(R) = \int_{\mathcal{R}_1(G')} \prod_{\nu=1}^{D} dy_\nu\ \theta\,(1 - \mathrm{Tr}\,y_\mu y_\mu)\,, \tag{2.28}$$

and we have absorbed the G' Weyl measure thus restoring the integral to G' gauge invariant form. Using the identity $\theta(1 - \mathrm{Tr}\,y_\mu y_\mu) = \int_0^1 dt\,\delta(t - \mathrm{Tr}\,y_\mu y_\mu)$ and then rescaling $t = [u/R]^{2-\eta}$ and $y_\mu = \tilde{y}_\mu[u/R]^{1-\eta/2}$ gives

$$\mathcal{F}_{D,G'}(R) = (2-\eta)R^{-(1-\eta/2)Dg'}\int_0^R du\,u^{(1-\eta/2)Dg'-1}\mathcal{I}_{D,G'}(u)\,. \tag{2.29}$$

We shall proceed by induction. Our aim is to show that

$$\int_0^\infty dR\,R^{Dg-1}\mathcal{I}_{D,G}(R) < \mathrm{const}\,. \tag{2.30}$$

If this is true for G', then 2.29 tells us

$$\mathcal{F}_{D,G'}(R) < B_2 R^{-(1-\eta/2)Dg'} \tag{2.31}$$

so that by (2.27)

$$\mathcal{I}_{D,G}(R) < B_3 R^{-(1-\eta/2)[2(D-1)(g-g'-1)+Dg']} \tag{2.32}$$

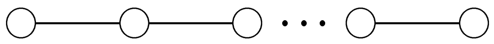
and we can then decide on the truth of (2.30) for G. Our task then is to find the regularly embedded subalgebras $\mathcal{G}'$ of $\mathcal{G}$ and choose the one which leads to the least inverse power in (2.32). In doing this we will need the result that if the regularly embedded subalgebra $\mathcal{G}'$ is a direct sum of two (mutually commuting) subalgebras $\mathcal{G}' = \mathcal{G}'_1 \oplus \mathcal{G}'_2$ then

$$\mathcal{F}_{D,G'}(R) < \mathcal{F}_{D,G'_1}(R)\, \mathcal{F}_{D,G'_2}(R) \tag{2.33}$$

since $\theta(1 - \mathrm{Tr}_{\mathcal{G}'}\, y_\mu y_\mu) \leq \theta(1 - \mathrm{Tr}_{\mathcal{G}'_1}\, y_\mu y_\mu)\, \theta(1 - \mathrm{Tr}_{\mathcal{G}'_2}\, y_\mu y_\mu)$ and $S_{\mathcal{G}'} = S_{\mathcal{G}'_1} + S_{\mathcal{G}'_2}$.

We now proceed to consider each group in turn:

SU$(r+1)$: The case of SU$(r+1)$ is dealt with in our previous paper [27]. We include a review here for completeness. The Dynkin diagram for $su(r+1)$ is



(2.34)

where there are r nodes. To find the regularly embedded subalgebras $\mathcal{G}'$ we simply remove one of the nodes, and discover

$$su(r+1) \to \mathcal{G}' = su(m) \oplus su(r+1-m)\,, \qquad 1 \leq m \leq r\,, \tag{2.35}$$

where we define $su(1) = 0$. The dimension of $su(m)$ is $m^2 - 1$, so that $g' = m^2 + (r+1-m)^2 - 2$.

The Lie algebra $su(2)$ has no regularly embedded subalgebra, so $g' = 0$ and

$$\mathcal{I}_{D,\mathrm{SU}(2)} < B_3 R^{-(1-\eta/2)4(D-1)}\,. \tag{2.36}$$

Then $\mathcal{Z}_{D,\mathrm{SU}(2)}$ is finite for $D \geq 5$. Substituting into (2.29), we see

$$\mathcal{F}_{D,\mathrm{SU}(2)} < B_4 R^{(1-\eta/2)3D} R^{(1-\eta/2)\delta_{D,3}} (\log R)^{\delta_{D,4}}\,. \tag{2.37}$$

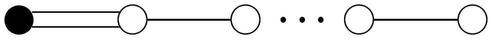
In dimensions 3 and 4, the result is at variance with (2.31). However, modification by a $\log R$ factor will not affect any of our conclusions, so it is only for $D = 3$ that we must be careful to use the modified formula.

The Lie algebra $su(3)$ has $su(2)$ as its only regularly embedded subalgebra. Then substituting (2.37) into (2.27), we discover $\mathcal{Z}_{D,\mathrm{SU}(3)}$ converges for $D \geq 4$. In this case, the general formula (2.31) is modified only in the case $D = 3$, and only by a factor of $\log R$ which will not affect our results.

For SU$(r+1)$ with $r \geq 3$, it is a simple exercise to discover which of these possible $\mathcal{G}'$ gives the least inverse power of R behaviour in (2.32). The only point to remember is that we must include an extra $R^{1-\eta/2}$ factor in the case of $\mathcal{G}' = su(2) \oplus su(r-1)$ when $D = 3$, to allow for the anomalous behaviour of $\mathcal{F}_{3,\mathrm{SU}(2)}$. We discover that we must always take $\mathcal{G}' = su(r)$, giving $g' = r^2 - 1$. Substituting back into (2.32), we find that $\mathcal{Z}_{D,\mathrm{SU}(r+1)}$ is convergent for $D \geq 3$ when $r \geq 3$. The correlation function (1.5) converges when $k < k_c$ with

$$k_c = 2rD - D - 4r - \delta_{D,3}\delta_{r,2}\,, \qquad r \geq 2, D \geq 3\,. \tag{2.38}$$

$\mathrm{SO}(2r+1)$**, r ≥ 2:** The Dynkin diagram for $so(2r+1)$ is



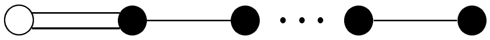
(2.39)

where there are r nodes, and the dimension is $g = 2r^2 + r$. By removing one node, we see that the possible $\mathcal{G}'$ are $so(2m+1) \oplus su(r-m)$ with $0 \leq m \leq r-1$. We discover the most important contribution is always from $\mathcal{G}' = so(2r-1)$, and that $\mathcal{Z}_{D,\mathrm{SO}(2r+1)}$ always converges for $r \geq 2$ and $D \geq 3$. The critical degree k_c for correlation functions is

$$\begin{aligned} &k_c = 2 && r = 2\,, D = 3 \\ &k_c = 4rD - 8r - 3D + 4 && \text{otherwise.} \end{aligned} \tag{2.40}$$

The exception when $r = 2$ and $D = 3$ occurs because of the anomalous behaviour of $\mathcal{F}_{3,\mathrm{SU}(2)}$.

Sp(2r), r ≥ 2: The Dynkin diagram for $sp(2r)$ is

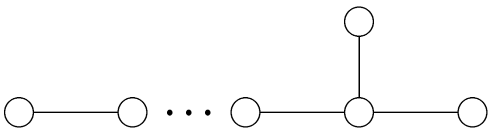


(2.41)

where there are r nodes, and the dimension is $g = 2r^2 + r$. The possible $\mathcal{G}'$ are $sp(2m) \oplus su(r-m)$ with $0 \leq m \leq r-1$, and the dominant contribution is from $sp(2r-2)$. The partition function $\mathcal{Z}_{D,Sp(2r)}$ converges for all $r \geq 2$ and $D \geq 3$ and the critical correlation function is given by

$$\begin{aligned} &k_c = 2 && r = 2\,, D = 3 \\ &k_c = 4rD - 8r - 3D + 4 && \text{otherwise.} \end{aligned} \tag{2.42}$$

$\mathrm{SO}(2r)$**, r ≥ 4:** The Dynkin diagram for $so(2r)$ is

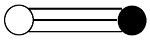


(2.43)

where there are r nodes, and the dimension is $g = 2r^2 - r$. The possible $\mathcal{G}'$ are $so(2m) \oplus su(r-m)$ for $4 \leq m \leq r-1$, $su(4) \oplus su(r-3)$, $su(r-2) \oplus su(2) \oplus su(2)$ and $su(r)$. The dominant contribution always comes from $so(2r-2)$, and we discover that $\mathcal{Z}_{D,\mathrm{SO}(2r)}$ always converges for $D \geq 3$ and $r \geq 4$. The critical correlation function is given by

$$k_c = 4rD - 5D - 8r + 8\,. \tag{2.44}$$

$\mathbf{G_2}$: The Dynkin diagram is

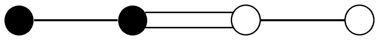


(2.45)

and the dimension is 14. The only regularly embedded subalgebra is $su(2)$, and we discover $\mathcal{Z}_{D,G_2}$ converges for $D \geq 3$ with

$$k_c = 9D - 20 - \delta_{D,3} \,. \tag{2.46}$$

$\mathbf{F_4}$: The Dynkin diagram is

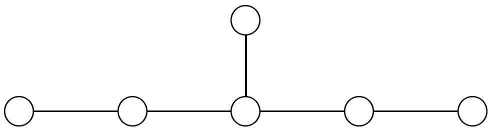


(2.47)

and the dimension $g = 52$. The dominant contributions come equally from $\mathcal{G}' = so(7)$ and $\mathcal{G}' = sp(6)$, each having $g' = 21$. Then $\mathcal{Z}_{D,F_4}$ converges for $D \geq 3$ and

$$k_c = 29D - 60 \,. \tag{2.48}$$

$\mathbf{E_6}$: The Dynkin diagram is

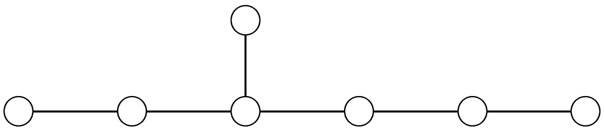


(2.49)

and the dimension $g = 78$. The dominant contribution comes from $\mathcal{G}' = so(10)$ having $g' = 45$. Then $\mathcal{Z}_{D,E_6}$ converges for $D \geq 3$ and

$$k_c = 31D - 64. \tag{2.50}$$

$\mathbf{E_7}$: The Dynkin diagram is

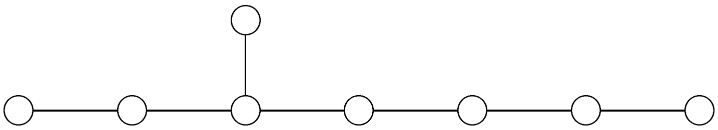


(2.51)

and the dimension $g = 133$. The dominant contribution comes from $\mathcal{G}' = e_7$ with $g' = 78$. Then $\mathcal{Z}_{D,E_7}$ converges for $D \geq 3$ and

$$k_c = 53D - 108 \,. \tag{2.52}$$

$\mathbf{E_8}$: The Dynkin diagram is



(2.53)

with dimension $g = 248$. The dominant contribution comes from $\mathcal{G}' = e_7$ with $g' = 133$. Then $\mathcal{Z}_{D,E_8}$ converges for $D \geq 3$ and

$$k_c = 113D - 228 \,. \tag{2.54}$$

3. Divergent bosonic integrals

The partition functions trivially diverge when $D = 2$ since, taking X_1 to be in the Cartan subalgebra, the integrand is independent of the Cartan subalgebra degrees of freedom of X_2. In [27] we showed that the SU(N) partition function diverges whenever the convergence conditions are not met. We will now show that the correlation function $\langle (\mathrm{Tr}\, X_\mu X_\mu)^{k/2} \rangle$ always diverges when $k \geq k_c$ so that k_c is indeed critical.

We have immediately that

$$\langle (\mathrm{Tr}\, X_\mu X_\mu)^{k/2} \rangle = \int_0^\infty dR R^{Dg+k-1} \mathcal{X}_{D,G}(R) \,. \tag{3.1}$$

This time, consider the region

$$\mathcal{R}: \;\; S < R^{-4} \tag{3.2}$$

Then $\exp(-R^4 S) > \exp(-1)$ and so

$$\mathcal{X}_{D,G}(R) > C_1 \mathcal{I}_{D,G} \,, \tag{3.3}$$

where now

$$\mathcal{I}_{D,G} = \int_{\mathcal{R}} \prod_{\nu=1}^{D} dx_\nu \; \delta\left(1 - \mathrm{Tr}\, x_\mu x_\mu\right) \tag{3.4}$$

and, moving x_1 into the Cartan subalgebra,

$$\mathcal{X}_{D,G}(R) > C_2 \int_{\mathcal{R}} \prod_{i=1}^{l} dx_1^i \Delta_G^2(x_1^i) \prod_{\nu=2}^{D} dx_\nu \; \delta\left(1 - \mathrm{Tr}\, x_\mu x_\mu\right). \tag{3.5}$$

Now pick a regularly embedded sub-algebra $\mathcal{G}'$ of $\mathcal{G}$ (with rank 1 less than $\mathcal{G}$). As before, write $x = y + \rho J + \omega^\beta F^\beta$ with $y \in \mathcal{G}'$, and define a new region $\mathcal{R}'_\epsilon$ by

$$\begin{aligned} \mathcal{R}'_\epsilon: \quad & \left|\omega_\nu^\beta\right| < \epsilon R^{-2} \qquad \nu = 2, \ldots, D \\ & S_{G'}(y_\mu) < \epsilon R^{-4} \,. \end{aligned} \tag{3.6}$$

Then by taking ϵ small enough, we see from 2.24 that

$$R'_\epsilon \subset \mathcal{R} \tag{3.7}$$

and therefore

$$\mathcal{X}_{D,G}(R) > C_2 \int_{\mathcal{R}'_\epsilon} \prod_{i=1}^{l} dx_1^i \Delta_G^2(x_1^i) \prod_{\nu=2}^{D} dx_\nu \; \delta\left(1 - \mathrm{Tr}\, x_\mu x_\mu\right). \tag{3.8}$$

Now perform this integral over the ω and ρ to discover

$$\mathcal{X}_{D,G}(R) > C_3 R^{-2(D-1)(g-g'-1)} \mathcal{F}_{D,G'}(R) \,, \tag{3.9}$$

where in this case

$$\begin{aligned}\mathcal{F}_{D,G}(R) &= \int_{\mathcal{R}} \prod_{\nu=1}^{D} dx_\nu\ \theta\left(1-\operatorname{Tr} x_\mu x_\mu\right)\left(1-\operatorname{Tr} x_\mu x_\mu\right)^{(D-1)/2} \\ &= C_4 R^{-Dg} \int_0^R du u^{Dg-1}\left(1-\frac{u}{R}\right)^{(D-1)/2} \mathcal{I}_{D,G}(u)\,. \end{aligned} \tag{3.10}$$

If $\mathcal{G}'$ does not contain $su(2)$ as an invariant subalgebra, then

$$\int_0^R du u^{Dg-1}\left(1-\frac{u}{R}\right)^{(D-1)/2} \mathcal{I}_{D,G}(u) > C_5 = \text{const} \qquad (\text{for } R>1)\,. \tag{3.11}$$

In the case of SU(2), we can repeat the argument leading to (3.9) (taking $G'=1$) to find $\mathcal{I}_{D,\mathrm{SU}(2)} > C_5 u^{-4(D-1)}$ for large u so that

$$\begin{aligned}\int_0^R du u^{Dg-1}\left(1-\frac{u}{R}\right)^{(D-1)/2} \mathcal{I}_{D,G}(u) &> C_6 && D\geq 5 \\ &> C_7 \log R && D=4 \\ &> C_8 R && D=3\,. \end{aligned} \tag{3.12}$$

Then

$$\mathcal{X}_{D,G}(R) > C_9 R^{-2(D-1)(g-g'-1)} R^{-Dg'} R^{\delta}, \tag{3.13}$$

where $\delta=1$ if $D=3$ and $G'=\mathrm{SU}(2)$ and zero otherwise, which is essentially the converse of (2.32). Finally the usual power counting argument leads to the result (note that this time we do not need to go through all the sub-algebras but just the one which gives the most divergent behaviour for the partition function).

4. Convergent supersymmetric integrals

We proceed as for the bosonic integrals to set

$$\mathcal{Z}_{D,G} = \int_0^\infty dR R^{Dg-1} R^{(D-2)g} \mathcal{X}_{D,G}(R)\,, \tag{4.1}$$

where now

$$\mathcal{X}_{D,G}(R) = \int \prod_{\nu=1}^{D} dx_\nu\, \mathcal{P}_{D,G}(x_\sigma)\, \delta\left(1-\operatorname{Tr} x_\mu x_\mu\right) \exp\left(-R^4 S_{\mathcal{G}}\right). \tag{4.2}$$

As before, it is sufficient to consider the region

$$\mathcal{R}_1(\mathcal{G}): \quad S_{\mathcal{G}} < R^{-2(2-\eta)}\,. \tag{4.3}$$

We shall again argue by induction, and for the induction step to work, we actually need to prove the result for the generalised pfaffian

$$[\mathcal{P}^r_{D,G}(x,R)]^{a_1,\dots,a_{2r}}_{\alpha_1,\dots,\alpha_{2r}} = R^{-(2-\eta)2r} \int d\psi \exp(\operatorname{Tr} \Gamma^\mu_{\alpha\beta}\psi_\alpha[x_\mu,\psi_\beta])\psi^{a_1}_{\alpha_1}\cdots\psi^{a_{2r}}_{\alpha_{2r}} \tag{4.4}$$

which exists for $r = 0, \ldots, (D-2)g$, and is modified from the usual definition by the inclusion of $2r$ fermionic insertions, each with an accompanying factor of $R^{-(2-\eta)}$. If we set $r = 0$ then the original pfaffian $\mathcal{P}_{D,G}$ is recovered (and is of course independent of R).

The structure of the Γs will be irrelevant from now on; their only relevant property, which we will use repeatedly, is that $\left|\Gamma^{\mu}_{\alpha\beta}\right| \leq 1$. For a more compact notation we shall suppress the dependence on Γ, and the explicit spinor and group indices, and write

$$\mathcal{P}^r_{D,G}(x,R) = R^{-(2-\eta)2r} \int d\psi \exp(\mathrm{Tr}\,\psi[x,\psi])\psi^1 \cdots \psi^{2r} \,. \tag{4.5}$$

Then defining

$$\mathcal{I}^r_{D,G}(R) = \int_{\mathcal{R}_1(\mathcal{G})} \prod_{\nu=1}^{D} dx_\nu \left|\mathcal{P}^r_{D,G}(x,R)\right| \delta\left(1 - \mathrm{Tr}\,x_\mu x_\mu\right) \tag{4.6}$$

we have

$$\left|\mathcal{X}^r_{D,G}(R)\right| < A_1 \exp(-R^{2\eta}) + \mathcal{I}^r_{D,G}(R) \,. \tag{4.7}$$

Proceeding as in the bosonic case, we choose the relevant regularly embedded subalgebra $\mathcal{G}'$, expand $x_\mu = y_\mu + \rho_\mu J + \omega^\beta_\mu F^\beta$, and note

$$\left|\omega^\beta_\nu\right| < c^{-1} D^{1/2} R^{-(2-\eta)} \,, \qquad \nu = 2, \ldots, D \,. \tag{4.8}$$

Further, write

$$\psi = \phi + \xi + \chi \tag{4.9}$$

with $\phi \in \mathcal{G}'$, $\xi = \xi J$ and $\chi = \chi^\beta F^\beta$. Using the relations (2.21), we find

$$\begin{aligned} \mathrm{Tr}\,\psi\,[x,\psi] \;=\; & \mathrm{Tr}\,\phi\,[y,\phi] + \mathrm{Tr}\,\phi\,[\omega,\chi] + \mathrm{Tr}\,\chi\,[\omega,\phi] + \\ & + \mathrm{Tr}\,\chi\,[\omega,\xi] + \mathrm{Tr}\,\xi\,[\omega,\chi] + \mathrm{Tr}\,\chi\,[x,\chi] \;, \end{aligned} \tag{4.10}$$

where $\rho = \rho J$ and $\omega = \omega^\beta F^\beta$. Inserting this expression into the definition (4.4), and expanding part of the exponential, we get

$$\begin{aligned} \mathcal{P}^r_{D,G}(x,R) = & \int d\phi d\chi d\xi \left(\frac{\xi^1 \cdots \xi^k}{R^{k(2-\eta)}} \frac{\phi^1 \cdots \phi^m}{R^{m(2-\eta)}} \frac{\chi^1 \cdots \chi^n}{R^{n(2-\eta)}} \right) \times \\ & \times \exp\left(\mathrm{Tr}\,\phi\,[y,\phi] + \mathrm{Tr}\,\phi\,[\omega,\chi] + \mathrm{Tr}\,\chi\,[\omega,\phi] + \mathrm{Tr}\,\chi\,[x,\chi]\right) \times \\ & \times \frac{1}{(2(D-2)-k)!} \left(\mathrm{Tr}\,\chi\,[\omega,\xi] + \mathrm{Tr}\,\xi\,[\omega,\chi]\right)^{2(D-2)-k} \;, \end{aligned} \tag{4.11}$$

where $k + m + n = 2r$. First we do the integrals over the $\mathcal{N} = 2(D-2)$ grassman variables ξ_α each of which is paired either with an ω, or with an explicit factor $R^{-(2-\eta)}$; since ω is itself bounded we get

$$\begin{aligned} \left|\mathcal{P}^r_{D,G}(x,R)\right| < \frac{R^{-2(D-2)(2-\eta)}}{(2(D-2)-k)!} \sum_P \Bigg| \int d\phi d\chi & \left(\frac{\phi^1 \cdots \phi^m}{R^{m(2-\eta)}} \frac{\chi^1 \cdots \chi^{n+2(D-2)-k}}{R^{n(2-\eta)}} \right) \times \\ & \times \exp\left(\mathrm{Tr}\,\phi\,[y,\phi] + \mathrm{Tr}\,\phi\,[\omega,\chi] + \mathrm{Tr}\,\chi\,[\omega,\phi] + \mathrm{Tr}\,\chi\,[x,\chi]\right) \Bigg| \,, \end{aligned} \tag{4.12}$$

where P simply indicates all the possible permutations of indices that can be generated. Next we expand the $\phi\omega\chi$ terms to get

$$\begin{aligned}\left|\mathcal{P}^r_{D,G}(x,R)\right| < R^{-2(D-2)(2-\eta)} &\sum_P \sum_l \frac{2^l}{l!(2(D-2)-k)!} \times \\ &\times \left| \int d\phi \left(\frac{\phi^1 \cdots \phi^{m+l}}{R^{(m+l)(2-\eta)}}\right) \exp\left(\mathrm{Tr}\, \phi\, [y,\phi]\right) \right| \times \\ &\times \max_x \left| \int d\chi \left(\frac{\chi^1 \cdots \chi^{n+2(D-2)-k+l}}{R^{n(2-\eta)}}\right) \exp\left(\mathrm{Tr}\, \chi\, [x,\chi]\right) \right| . \end{aligned} \tag{4.13}$$

Finally we integrate out the χ fermions and use the fact that x is a bounded quantity to obtain

$$\left|\mathcal{P}^r_{D,G}(x,R)\right| < R^{-(2-\eta)2(D-2)} \sum_{r'} C_{r'} \left|\mathcal{P}^{r'}_{D,G'}(y,R)\right| , \tag{4.14}$$

where the $C_{r'}$ are constants. In the spirit of the notation (4.5), we have suppressed sums over the many possible combinations of indices.

Inserting the bound (4.14) into (4.6) we get

$$\mathcal{I}^r_{D,G}(R) < R^{-(2-\eta)2(D-2)} \sum_{r'} C_{r'} \int_{\mathcal{R}_1} \prod_{\nu=1}^{D} dx_\nu \left|\mathcal{P}^{r'}_{D,G'}(y,R)\right| \delta\left(1 - \mathrm{Tr}\, x_\mu x_\mu\right) \tag{4.15}$$

and we can now follow the bosonic procedure and integrate out the ω and ρ degrees of freedom to obtain

$$\begin{aligned}\mathcal{I}^r_{D,G}(R) < R^{-(2-\eta)2(D-2)} &R^{-(2-\eta)(D-1)(g-g'-1)} \times \\ &\times \sum_{r'} C_{r'} \int_{\mathcal{R}_1(G')} \prod_{\nu=1}^{D} dy_\nu \left|\mathcal{P}^{r'}_{D,G'}(y,R)\right| \theta\left(1 - \mathrm{Tr}\, y_\mu y_\mu\right). \end{aligned} \tag{4.16}$$

As before, replace $\theta(1 - \mathrm{Tr}\, y_\mu y_\mu) = \int_0^1 dt\, \delta(t - y_\mu y_\mu)$ and rescale $t = [u/R]^{2-\eta}$ and $y_\mu = \tilde{y}_\mu [u/R]^{1-\eta/2}$ giving

$$\begin{aligned}\mathcal{I}^r_{D,G}(R) < R^{-(2-\eta)2(D-2)} &R^{-(2-\eta)(D-1)(g-g'-1)} \times \\ &\times \sum_{r'} C_{r'} \int_0^R \frac{du}{u} [u/R]^{(2-\eta)[(D-1)g'+3r'/2]} \times \\ &\times \int_{\mathcal{R}_1(G')} \prod_{\nu=1}^{D} d\tilde{y}_\nu \left|\mathcal{P}^{r'}_{D,G'}(\tilde{y},u)\right| \delta\left(1 - \mathrm{Tr}\, \tilde{y}_\mu \tilde{y}_\mu\right). \end{aligned} \tag{4.17}$$

Since $u/R < 1$, this can be reduced to

$$\mathcal{I}^r_{D,G}(R) < R^{-(2-\eta)[2(D-2)+(D-1)(g-1)]} \sum_{r'} C_{r'} \int_0^R \frac{du}{u} u^{(2-\eta)(D-1)g'} \mathcal{I}^{r'}_{D,G'} . \tag{4.18}$$

We argue by induction, so assume that, for G'

$$\int_0^\infty dR R^{Dg'-1} R^{(D-2)g'} \mathcal{I}^r_{D,G'}(R) \tag{4.19}$$

converges for $D > 3$, and all choices of r. Then (4.18) gives

$$\mathcal{I}^r_{D,G}(R) < CR^{-(2-\eta)[2(D-2)+(D-1)(g-1)]} \tag{4.20}$$

and in particular, the induction statement is true also for G. It remains to check that the induction statement is true for the smallest possible regularly embedded subalgebra, which is $su(2)$. Since $su(2)$ has no regularly embedded subalgebra, we can repeat the above arguments with $\mathcal{G}' = 0$ and find

$$\mathcal{I}^r_{D,\mathrm{SU}(2)}(R) < CR^{-(2-\eta)[2(D-2)+2(D-1)]} \tag{4.21}$$

and this completes the proof.

Taking now $r = 0$, we have discovered that, for any compact semi-simple group G,

$$\mathcal{I}_{D,G}(R) < CR^{-(2-\eta)[2(D-2)+(D-1)(g-1)]} \tag{4.22}$$

and in particular, the partition function $\mathcal{Z}_{D,G}$ converges for $D > 3$. It is a remarkable fact that the bound (4.22) does not depend on the sub-algebra. For the correlation function (1.5) to converge, we require

$$Dg + (D-2)g + k < 2[2(D-2) + (D-1)(g-1)] \tag{4.23}$$

and so the critical value is

$$k_c = 2(D-3) \tag{4.24}$$

independently of the gauge group.

5. Discussion

For the bosonic theories, we have shown that the partition function converges when $D \geq D_c$ and calculated D_c for each of the compact simple groups. It is a simple exercise to extend the result to the compact semi-simple groups since they are built out of the simple groups. For example, $so(4) = su(2) \oplus su(2)$, so $\mathcal{Z}_{D,\mathrm{SO}(4)}$ converges when $D \geq D_c = 5$. In addition, we have calculated the critical degree k_c for correlation functions, such that $\langle \mathcal{C}_k \rangle$ converges when $k < k_c$. Conversely, we have shown that there always exists a correlation function of degree k_c which diverges.

It happens that, for the simple groups, the result D_c is equal to the result one would obtain by assuming every $(x \cdot \alpha)^2$ is bigger than a constant in equation (2.18) (the one loop approximation). Thus, in this case, there is a simple rule

$$D > \frac{2(g-l)}{g-2l} \tag{5.1}$$

for the partition function to converge. However, this rule fails for the semi-simple groups (as we quickly see by considering $su(r+1) \oplus su(2)$ for large r). In general, the one loop approximation gives the wrong value for k_c (except in the case of $su(2)$ where it is exact).

We have shown that the supersymmetric partition function converges in $D = 4, 6$ and 10 with any compact semi-simple gauge group, and that correlation functions of degree

$$k < k_c = 2(D - 3) \tag{5.2}$$

are convergent independent of the gauge group. In the case of $\mathrm{SU}(r+1)$, this result corresponds to a conjecture [42] based on Monte Carlo evaluation of the integrals for small r. We have not found rigorous lower bounds in the supersymmetric case so it remains unproven that k_c is critical; however, there can be little doubt that, for example, $\langle \mathrm{Tr}\, X_\mu^2 \rangle$ is logarithmically divergent for $D = 4$. The situation for partition functions in $D = 3$ may be more complicated; the integrals are absolutely divergent but for some groups the pfaffian is an odd function of X_μ so that the integral vanishes naively. Only if there were a supersymmetric regularization of the integrals would it be possible that some of these theories make sense.

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